Lagrangians and Hamiltonians for High School Students

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Abstract

A discussion of Lagrangian and Hamiltonian dynamics is presented at a level which should be suitable for advanced high school students. This is intended for those who wish to explore a version of mechanics beyond the usual Newtonian treatment in high schools, but yet who do not have advanced mathematical skills.

1 Introduction

Newtonian dynamics is usually taught in high school physics courses and in college level freshman physics class [1]. Lagrangian and Hamiltonian dynamics [2, 3] is usually reserved for an upper division undergraduate physics course on classical dynamics. This is all as it should be, particularly since one needs the technique of calculus of variations for the Lagrangian formulation.

However it is always nice to be able to whet the appetite of the advanced high school student for a taste of things to come. For those students who have successfully mastered the contents of the typical high school physics course, one can give an extra lesson on Lagrangian and Hamiltonian dynamics without having to use calculus of variations. The idea is simply to present some new formulations of dynamics that an advanced high school student will find enjoyable and intellectually interesting. (The students can be told that a rigorous formulation will be presented in college courses.)

For simplicity, consider only the one-dimensional problem. Write Newton's equation

$$F = ma \tag{1}$$

and define the potential energy $U(x)$, which is a function only of position, as

$$F \equiv -\frac{dU}{dx} \tag{2}$$

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where $-\frac{dU}{dx}$ is the spatial derivative of the potential energy. Thus re-write Newton's equation as

$$-\frac{dU}{dx} = m\ddot{x} \tag{3}$$

where $\dot{x} \equiv \frac{dx}{dt} = v$ for the speed and $\ddot{x} \equiv \frac{d^2x}{dt^2} = a$ for the acceleration.

2 Lagrangian Dynamics

To introduce Lagrangian dynamics define a *Lagrangian* as a function of the two variables of position x and speed $\dot{x}$

$$L(x, \dot{x}) \equiv T(\dot{x}) - U(x) = \frac{1}{2}m\dot{x}^2 - U(x) \tag{4}$$

where the kinetic energy $T(\dot{x}) \equiv \frac{1}{2}m\dot{x}^2$ is a function only of the speed variable and the potential energy again is only a function of position $U(x)$.

Now introduce the idea of a partial derivative. This is very easy. For a function of a single variable $f(y)$ the notation $\frac{df}{dy}$ is used for the derivative. For a function of *two* variables $g(y, z)$ there are two possible derivatives for each variable y or z. In this case one simply introduces a different notation for derivative, namely $\frac{\partial g}{\partial y}$ for the y derivative (where y is changing but z is constant) and $\frac{\partial g}{\partial z}$ for the z derivative (where z is changing but y is constant). Even though high school students won't see partial derivatives until they are in college, nevertheless the idea is very simple and can easily be explained to the advanced student who is taking a course in calculus.

From (4) one can easily see that

$$\frac{\partial L}{\partial x} = -\frac{dU}{dx} \tag{5}$$

and

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \equiv p \tag{6}$$

which is called the momentum p. Obviously then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \tag{7}$$

Combining (5), (7) and (3), Newton's equation (3) becomes

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \tag{8}$$

which is the Euler-Lagrange equation in one dimension. It can be explained to the students that it is this equation in Lagrangian dynamics which replaces $F = ma$ in Newtonian dynamics.

2.1   Lagrangian example

Students will obviously want to see some examples of how the Lagrangian formulation works. A simple example is the one-dimensional harmonic oscillator with

$$F \equiv -\frac{dU}{dx} = -kx. \tag{9}$$

Newton's equation is

$$- kx = m\ddot{x}. \tag{10}$$

The potential $U(x)$ is obtained by integrating (9) to give

$$U(x) = \frac{1}{2}kx^2. \tag{11}$$

Thus the Lagrangian is

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{12}$$

giving

$$\frac{\partial L}{\partial x} = -kx \tag{13}$$

and substituting into (8) and (7) gives exactly back the same equation of motion (10) as in the Newtonian case.

Many teachers will have had the students work out the equation of motion from Newtonian dynamics for other types of forces, such as a particle in a uniform gravitational field. Students can be encouraged to prove that the same equations of motion result from the Lagrangian formulation. Students can also be encouraged to think about three-dimensional problems and to derive, on their own, the three Euler-Lagrange equations (corresponding to

the three component equations $F_x = m\ddot{x}$, $F_y = m\ddot{y}$, $F_z = m\ddot{z}$) which result from the three dimensional Lagrangian

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z). \qquad (14)$$

3 Hamiltonian Dynamics

Now consider the Hamiltonian formulation of dynamics. Define a *Hamiltonian* as a function of the two variables, momentum p and position x,

$$H(p, x) \equiv p\dot{x} - L(x, \dot{x}) \qquad (15)$$

which can be seen to be just the total energy $T + U$ as $H = p\dot{x} - L = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + U = \frac{1}{2}m\dot{x}^2 + U = T + U$. Hamilton's equations follow immediately. L is not a function of p and therefore

$$\frac{\partial H}{\partial p} = \dot{x}. \qquad (16)$$

But L is a function of x and thus

$$\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x}. \qquad (17)$$

However (6) and (8) give $\frac{\partial L}{\partial x} = \dot{p}$ so that

$$-\frac{\partial H}{\partial x} = \dot{p}. \qquad (18)$$

Equation (16) and (18) are Hamilton's equations which replace $F = ma$ in Newtonian dynamics.

3.1 Hamiltonian example

For the harmonic oscillator example, the Hamiltonian is

$$H(p, x) = p\dot{x} - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}kx^2 \qquad (19)$$

where we have had to replace $\dot{x}$ by $\frac{p}{m}$ because $H(p,x)$ is supposed to be a function of p and x only. Thus Hamilton's equations (16) and (18) give

$$\frac{p}{m} = \dot{x} \tag{20}$$

and

$$-kx = \dot{p}. \tag{21}$$

These are shown to give the equation of motion (10) by differentiating (20) as

$$\frac{\dot{p}}{m} = \ddot{x} \tag{22}$$

and substituting (21) for $\dot{p}$ gives back equation (10).

Once again students can be encouraged to use other examples that they have already studied in Newtonian dynamics and to show that Hamilton's equations result in the same equation of motion. Again students can work out the three-dimensional generalization of Hamilton's equations using

$$H(p_x, p_y, p_z, x, y, z) = p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - L(x, y, z, \dot{x}, \dot{y}, \dot{z}). \tag{23}$$

Finally teachers can emphasize to students that Newtonian mechanics is based on forces, whereas Lagrangian and Hamiltonian dynamics is based on energy.

In summary, a discussion of Lagrangian and Hamiltonian dynamics has been presented which should be suitable for advanced high school students, who are interested in exploring some topics not normally presented in the high school physics curriculum. It is also hoped that this article can be given to students to read on their own.

References

[1] R. A. Serway, *Principles of Physics*, (Saunders, New York, 1998), pp. 80-141.

[2] T. L. Chow, *Classical Mechanics*, (Wiley, New York, 1995), pp. 99-175.

[3] G. R. Fowles and G. L. Cassiday, *Analytical Mechanics*, 5th ed., (Saunders, New York, 1993), pp. 340-373.